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The Fekete-Szegö Theorem for a Certain Class of Analytic Functions

(Teorem Fekete-Szegö Bagi Suatu Kelas Fungsi Analisis)

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ABSTRACT

In this paper, we discuss a well known class studied by many authors including Ramesha et al. and Janteng, few to mention. Next, we extend the class to a wider class of functions f denoted by u_α^β, *which are normalized and univalent, in the open unit disk* $D=\{z:|z|<1\}$ *satisfying the condition:*

$$Re\left(\frac{a\,z^2 f''(z)}{g(z)}+\frac{z\,f'(z)}{g(z)}\right)>0, \qquad 0\le\alpha<1,$$

where $g\in S^*(\beta), g(z)\neq 0$ *is a normalized starlike function of order* β, *for* $0\le\beta<1$. *For* $f\in u_\alpha^\beta$ *we shall obtain sharp upper bounds for the Fekete-Szegö functional* $|a_3-\mu a_2^2|$ *when* μ *is real.*

Keywords: Close-to-convex functions; convex functions; Fekete- Szegö theorem; starlike functions; univalent functions

ABSTRAK

Dalam kertas kerja ini, kelas yang terkenal yang dikaji oleh ramai penulis akan dibincangkan, termasuklah beberapa nama seperti Ramesha et al. dan Janteng. Seterusnya, kelas ini diperluaskan kepada kelas fungsi f dilambangkan oleh u_α^β, *yang ternormal dan univalen dalam cakera unit terbuka* $D=\{z:|z|<1\}$ *memenuhi syarat:*

$$Re\left(\frac{a\,z^2 f''(z)}{g(z)}+\frac{z\,f'(z)}{g(z)}\right)>0, \qquad 0\le\alpha<1,$$

yang $g\in S^*(\beta), g(z)\neq 0$ *adalah fungsi bak-bintang ternormal peringkat* β, $(0\le\beta<1)$. *Untuk* $f\in u_\alpha^\beta$ *batas atas terbaik bagi fungsian Fekete-Szegö* $|a_3-\mu a_2^2|$ *diperoleh apabila* μ *adalah nyata.*

Kata kunci: Fungsi bak-bintang; fungsi cembung; fungsi hampir cembung; fungsi univalen; teorem Fekete- Szegö

INTRODUCTION AND DEFINITION

Let S denotes the class of normalized analytic univalent functions f of the form:

$$f(z)=z+\sum_{n=2}^{\infty}a_n z^n, \tag{1}$$

a_n is a complex number and $z\in D=\{z:|z|<1\}$

A classical theorem of Fekete & Szegö (1933), states that for $f\in S$ and given by (1),

$$\left|a_3-\mu a_2^2\right|\le 1+2\exp\left(\frac{-2\mu}{1-\mu}\right),$$

for $0\le\mu\le 1$and the inequality is sharp.

For the subclass of S, consisting of convex functions C, starlike functions S^*, and close-to-convex functions K. Sharp upper bounds for the functional $|a_3-\mu a_2^2|$ have been studied by various authors, however we mention only few names here such as Keogh and Merkes (1969), Koepf (1987), Darus and Thomas (1996, 2000), Ibrahim and Darus (2001), Rahman and Darus (2001), Darus (2002), Darus and Tuneski (2003), Darus and Hong (2004), and Frasin and Darus (2003).

In particular, for $f\in K$ and be given by (1), Keogh and Merkes (1969) showed that:

$$\left|a_3-\mu a_2^2\right|\le\begin{cases}3-4\mu, & \text{if } \mu\le\frac{1}{3},\\ \frac{1}{3}+\frac{4}{9\mu}, & \text{if } \frac{1}{3}\le\mu\le\frac{2}{3},\\ 1, & \text{if } \frac{2}{3}\le\mu\le 1,\\ 4\mu-3, & \text{if } \mu\ge 1,\end{cases}$$

and for each μ there is a function in K for which equality holds.

In this paper, we give an estimate for the same functional for the class u_α^β defined as follows:

Definition 1.1: For $0\le\alpha<1$ and $0\le\beta<1$, let the function f be analytic in D and be given by (1). Then the function $f\in u_\alpha^\beta$ if and only if there exist $g\in S^*(\beta)$ such that for $z\in D$

$$Re\left(\frac{az^2f''(z)}{g(z)}+\frac{zf'(z)}{g(z)}\right)>0. \tag{2}$$

Here, $S^*(\beta)$ denotes the class of starlike functions of order β, that is $g\in S^*(\beta)$ if and only if g is analytic in D and,

$$Re\left(\frac{zg'(z)}{g(z)}\right)>\beta,\quad g(z)\neq 0 \quad \text{for } z\in D. \tag{3}$$

Note that if the function $g(z)$ in (2) is replaced by $f(z)$, we get the definition of Ramesha et al. (1995). However, the authors (Ramesha et al.) did not define the inequality as in (3). The work of Ramesha et al. (1995) has attracted many authors such as Obradovic and Joshi (1998), Janteng (1999, 2006) and Akbarally (2001).

We first state some preliminary lemmas, required for proving our result.

Preliminary Results

Lemma 2.1: (Pommerenke 1975). Let h be analytic in D with $Re\, h(z)>0$ and be given by,

$h(z)=1+c_1z+c_2z^2+\ldots$, for $z\in D$, then

$|c_n|\le 2$, where $n\ge 1$,

and

$$\left|c_2-\frac{c_1^2}{2}\right|\le 2-\frac{|c_1|^2}{2}.$$

Lemma 2.2: (Darus & Thomas 1996). For $0\le\beta<1$, let $g\in S^*(\beta)$ with $g(z)=z+b_2z^2+b_3z^3\ldots$. Then, for μ real

$$|b_3-\mu b_2^2|\le(1-\beta)\max\{1,|3-2\beta-4\mu(1-\beta)|\}.$$

Next, we give our first result as follows:

Lemma 2.3: Let the function f given by (1) belongs to the class u_α^β. Then

$$(\alpha+1)|a_2|\le 2-\beta$$

and

$$3(2\alpha+1)|a_3|\le(3-2\beta)(3-\beta).$$

Proof. Since $g\in S^*(\beta)$, it follows from (3) that

$$zg'(z)=[\beta+(1-\beta)\,p(z)]\,g(z). \tag{4}$$

For $z\in\beta$, with $Re\, p(z)<0$ given by where $p_1, p_2, \ldots$, are complex numbers. Equating coefficients, we obtain

$$b_2=p_1(1-\beta), \tag{5}$$

and

$$2b_3=(p_2+b_2p_1)(1-\beta). \tag{6}$$

It also follows from (2) that

$$\alpha\, z^2f''(z)+zf'(z)=g(z)\,h(z) \tag{7}$$

where $Re\, h(z)>0$, writing $h(z)=1+c_1z+c_2z^2\ldots$, where $c_1, c_2, \ldots$, are complex numbers and from (7), we have

$$z+\sum_{n=2}^{\infty}a_nz^n\left(\alpha n^2-\alpha n+n\right)$$
$$=z+\left(c_1+b_2\right)z^2+\left(c_2+b_2c_1+b_3\right)z^3+\ldots.$$

And equating coefficients, gives

$$2(\alpha+1)a_2=c_1+b_2 \tag{8}$$

and

$$3(2\alpha+1)\,a_3=c_2+b_2c_1+b_3. \tag{9}$$

The result now follows on using classical inequalities, $|p_1|\le 2$, $|p_2|\le 2$, $|c_1|\le 2$, $|c_2|\le 2$, and applying the following inequalities,

$$|b_2|\le 2\,(1-\beta)$$

and

$$|b_3|\le(1-\beta)\,(3-2\beta),$$

which follow from (5) and (6).

Now, we shall show our main result for the class u_α^β.

Main Result

Theorem 3.1: Let the function f be given by (1) and belongs to the class u_α^β. Then, for $0\le\alpha<1$ and $0\le\beta<1$,

$$3(2\alpha+1)(a+1)\,2\,|a_3-u\,a_2^2|\le$$

$$\begin{cases}(\alpha+1)^2(3-2\beta)-3(2\alpha+1)(2-\beta)^2\mu, \\ \textit{if } \mu\le\dfrac{2(1-\beta)(\alpha+1)^2}{3(2-\beta)(2\alpha+1)} \\ (\alpha+1)^2\left(1+\beta(3-2\beta)+\dfrac{4(1-\beta)^2(\alpha+1)^2}{3(2\alpha+1)\mu}\right), \\ \textit{if } \dfrac{2(1-\beta)(\alpha+1)^2}{3(2-\beta)(2\alpha+1)}\le\mu\le\dfrac{2(\alpha+1)^2}{3(2\alpha+1)} \\ (\alpha+1)^2(3-\beta), \\ \textit{if } \dfrac{2(\alpha+1)^2}{3(2\alpha+1)}\le\mu\le\dfrac{2(3-\beta)(\alpha+1)^2}{3(2-\beta)(2\alpha+1)} \\ -(\alpha+1)^2(3-2\beta)(3-\beta)+3(2\alpha+1)(2-\beta)^2\mu \\ \textit{if } \dfrac{2(3-\beta)(\alpha+1)^2}{3(2-\beta)(2\alpha+1)}\le\mu.\end{cases}$$

Inequalities are sharp for all cases.

Proof. Write:

$$3(2\alpha+1)(a_3-\mu a_2^2)=3(2\alpha+1)a_3-3(2\alpha+1)\mu a_2^2.$$

From (8) and (9), we have:

$$3(2\alpha+1)(a_3-\mu a_2^2)=c_2+b_2c_1+b_3-3(2\alpha+1)\mu\left[\frac{c_1^2+b_2^2+2c_1b_2}{4(\alpha+1)^2}\right]$$

Then we get:

$$3(2\alpha+1)(a_3-\mu a_2^2)=b_3-\frac{3(2\alpha+1)\mu b_2^2}{4(\alpha+1)^2}+c_2$$
$$+c_1^2\left[\frac{2(\alpha+1)^2-3(2+1)\mu}{4(\alpha+1)^2}-\frac{1}{2}\right]$$
$$+b_2c_1\left[1-\frac{3(2\alpha+1)\mu}{2(\alpha+1)^2}\right]. \tag{10}$$

From (10) gives:

$$3(2\alpha+1)|a_3-\mu a_2^2|\le\left|b_3-\frac{3(2\alpha+1)\mu b_2^2}{4(\alpha+1)^2}\right|+\left|c_2-\frac{1}{2}c_1^2\right|$$
$$+\frac{1}{4(\alpha+1)^2}\left|2(\alpha+1)^2-3(2\alpha+1)\mu\right||c_1|^2$$
$$+\frac{|b_2||c_1|}{2(\alpha+1)^2}\left|2(\alpha+1)^2-3(2\alpha+1)\mu\right|.$$

Now consider the first case for all

$$\mu\le\frac{2(\alpha+1)^2}{3(2\alpha+1)}.$$

Having,

$$b_3-\frac{3(2\alpha+1)\mu b_2^2}{4(\alpha+1)^2}>0$$

and $2(\alpha+1)^2-3(2\alpha+1)\mu\ge 0$, and by applying the following inequalities,

$|b_2|\le 2(1-\beta)$, $|b_3|\le(3-2\beta)(1-\beta)$

and Lemma 2.1 we get:

$$3(2\alpha+1)|a_3-\mu a_2^2|$$
$$\le\left((3-2\beta)(1-\beta)+2-\frac{3(2\alpha+1)\mu(1-\beta)^2}{(\alpha+1)^2}\right)$$
$$-\frac{3(2\alpha+1)\mu|c_1|^2}{4(\alpha+1)^2}+\frac{(1-\beta)\left[2(\alpha+1)^2-3(2\alpha+1)\mu\right]|c_1|}{(\alpha+1)^2} \tag{11}$$

$=\phi(x)$, say, with $x=|c_1|$.

Elementary calculation indicates that the function ϕ attains its maximum value at:

$$x_o=\frac{2(1-\beta)\left[2(\alpha+1)^2-3(2\alpha+1)\mu\right]}{3(2\alpha+1)\mu}.$$

A straightforward calculation, we find:

$$\phi(x_o)=1+\beta(3-2\beta)+\frac{4(1-\beta)^2(\alpha+1)^2}{3(2\alpha+1)\mu}.$$

Now we have:

$$3(2\alpha+1)(\alpha+1)^2\left|a_3-\mu a_2^2\right|\le$$
$$(\alpha+1)^2\phi(x_o)=(\alpha+1)^2\left(1+\beta(3-2\beta)+\frac{4(1-\beta)^2(\alpha+1)^2}{3(2\alpha+1)\mu}\right). \tag{12}$$

Next, since $|x_o|\le 2$, we get, the interval $\frac{2(1-\beta)(\alpha+1)^2}{3(2-\beta)(2\alpha+1)}\le\mu$ in the case

$$\mu\le\frac{2(\alpha+1)^2}{3(2\alpha+1)}.$$

And hence, the result (12) completes the proof for the case:

$$\frac{2(1-\beta)(\alpha+1)^2}{3(2-\beta)(2\alpha+1)}\le\mu\le\frac{2(\alpha+1)^2}{3(2\alpha+1)}.$$

We can see that the result is sharp by choosing:

$$c_1=\frac{2(1-\beta)\left[2(\alpha+1)^2-3(2\alpha+1)\mu\right]}{3(2\alpha+1)\mu},$$

$c_2=2$, $p_1=2$, $p_2=2$, $b_2=2(1-\beta)$ and $b_3=(3-2\beta)(1-\beta)$, and substituting in (10) to get the desired result.

Secondly, we consider the case:

$$\mu\le\frac{2(1-\beta)(\alpha+1)^2}{3(2-\beta)(2\alpha+1)}.$$

Write,

$$a_3-\mu a_2^2=a_3-\frac{2(1-\beta)(\alpha+1)^2}{3(2-\beta)(2\alpha+1)}a_2^2+\left[\frac{2(1-\beta)(\alpha+1)^2}{3(2-\beta)(2\alpha+1)}-\mu\right]a_2^2.$$

Now we say that, by using the inequality $|a_2|\le\frac{2-\beta}{\alpha+1}$ we obtain the following:

$$3(2\alpha+1)(\alpha+1)^2|a_3-\mu a_2^2|$$
$$\le 3(2\alpha+1)(\alpha+1)^2\left|a_3-\frac{2(1-\beta)(\alpha+1)^2}{3(2-\beta)(2\alpha+1)}a_2^2\right|$$
$$+3(2\alpha+1)(\alpha+1)^2\left|\frac{2(1-\beta)(\alpha+1)^2}{3(2-\beta)(2\alpha+1)}-\mu\right||a_2|^2$$

$$\leq (\alpha+1)^2(5-3\beta)+$$
$$3(2\alpha+1)(\alpha+1)^2\left(\frac{2(1-\beta)(\alpha+1)^2}{3(2-\beta)(2\alpha+1)}-\mu\right)\frac{(2-\beta)^2}{(\alpha+1)^2}$$
$$=(\alpha+1)^2(3-2\beta)(3-\beta)-3(2\alpha+1)(2-\beta)^2\mu. \tag{13}$$

This completes the proof for this case.

Here, in order to find the upper bound for:

$$3(2\alpha+1)(\alpha+1)^2\left|a_3-\frac{2(1-\beta)(\alpha+1)^2}{3(2-\beta)(2\alpha+1)}a_2{}^2\right|,$$

we use the previous result of (12), which is already proved for:

$$\mu = \frac{2(1-\beta)(\alpha+1)^2}{3(2-\beta)(2\alpha+1)}.$$

After that, we have to make sure that the result is sharp for this case. Upon choosing, $c_1 = c_2 = p_1 = p_2$, $b_2 = 2(1-\beta)$ and b3 = $(3-2\beta)(1-\beta)$ in (10) we get the required result.

Next we consider the case for all:

$$\frac{2(\alpha+1)^2}{3(2\alpha+1)} \leq \mu.$$

Consider the case:

$$\frac{2(\alpha+1)^2}{3(2\alpha+1)} \leq \mu \leq \frac{2(3-\beta)(\alpha+1)^2}{3(2-\beta)(2\alpha+1)}.$$

First, we take the case:

$$\mu=\frac{2(3-\beta)(\alpha+1)^2}{3(2-\beta)(2\alpha+1)}.$$

We start from, (11) by substituting:

$$\mu=\frac{2(3-\beta)(\alpha+1)^2}{3(2-\beta)(2\alpha+1)}.$$

We get:

$$3(2\alpha+1)\left|a_3-\frac{2(3-\beta)(\alpha+1)^2}{3(2-\beta)(2\alpha+1)}a_2{}^2\right|$$
$$\leq \frac{-\beta^2-\beta+4}{(2-\beta)}-\frac{(3-\beta)|c_1|^2}{2(2-\beta)}-\frac{2(1-\beta)|c_1|}{(2-\beta)}$$

$= \psi(x)$, say, with $x = |c_1|$.

Assume our maximum value, obtained in $x_o \in [0, 2]$ but through calculation it happens on $x_\circ = \dfrac{-2(1-\beta)}{3-\beta}$ so it is out of the interval, so it contradicts the assumption. Then the maximum value may occur on the boundary $x_o = 0$ or 2. A straightforward calculation shows that the function $\psi(x)$ attains absolute maximum value in the interval [0, 2] at $x_o = 0$. And after some simplifications, we conclude that $\psi(x), \leq 3-\beta$, at $x_o = 0$.

And therefore, we get the result:

$$3(2\alpha+1)(\alpha+1)^2\left|a_3-\frac{2(3-\beta)(\alpha+1)^2}{3(2-\beta)(2\alpha+1)}a_2{}^2\right| \leq (\alpha+1)^2(3-\beta). \tag{14}$$

When $\mu=\dfrac{2(\alpha+1)^2}{3(2\alpha+1)}$ and substitute in last result (12) gives the following:

$$3(2\alpha+1)(\alpha+1)^2\left|a_3-\frac{2(3-\beta)(\alpha+1)^2}{3(2-\beta)(2\alpha+1)}a_2{}^2\right| \leq (\alpha+1)^2(3-\beta).$$

Hence completing the proof for the particular case.

$$\frac{2(\alpha+1)^2}{3(2\alpha+1)} \leq \mu \leq \frac{2(3-\beta)(\alpha+1)^2}{3(2-\beta)(2\alpha+1)}.$$

The result is sharp on choosing $c_1 = p_1 = 0$, $p_2 = 2$, $c_2 = 2$, $b_2 = 0$ and $b_3 = 1-\beta$ substituting in (10), and doing a simple calculation, we shall get the desired equality.

Finally, consider:

$$\frac{2(3-\beta)(\alpha+1)^2}{3(2-\beta)(2\alpha+1)} \leq \mu.$$

Write:

$$a_3-\mu a_2{}^2 = a_3-\frac{2(3-\beta)(\alpha+1)^2}{3(2-\beta)(2\alpha+1)}a_2{}^2 + \left[\frac{2(3-\beta)(\alpha+1)^2}{3(2-\beta)(2\alpha+1)}-\mu\right]a_2{}^2.$$

Now we say that, by using the inequality $|a_2| \leq \dfrac{2-\beta}{\alpha+1}$ we obtain the following:

$$3(2\alpha+1)(\alpha+1)^2\left|a_3-\mu a_2{}^2\right| \leq 3(2\alpha+1)(\alpha+1)^2\left|a_3-\frac{2(3-\beta)(\alpha+1)^2}{3(2-\beta)(2\alpha+1)}a_2{}^2\right|$$
$$+3(2\alpha+1)(\alpha+1)^2\left|\frac{2(3-\beta)(\alpha+1)^2}{3(2-\beta)(2\alpha+1)}-\mu\right||a_2|^2$$
$$\leq (\alpha+1)^2(3-\beta)+3(2\alpha+1)(\alpha+1)^2\left(\mu-\frac{2(3-\beta)(\alpha+1)^2}{3(2-\beta)(2\alpha+1)}\right)\frac{(2-\beta)^2}{(\alpha+1)^2}$$
$$=-(\alpha+1)^2(3-2\beta)(3-\beta)+3(2\alpha+1)(2-\beta)^2\mu. \tag{15}$$

Hence the proof is complete for this case.

Here,

$$3(2\alpha+1)(\alpha+1)^2\left|a_3-\frac{2(3-\beta)(\alpha+1)^2}{3(2-\beta)(2\alpha+1)}a_2{}^2\right| \leq (\alpha+1)^2(3-\beta),$$

we used the last result (14) already proven for

$$\mu = \frac{2(3-\beta)(\alpha+1)^2}{3(2-\beta)(2\alpha+1)}.$$

The result is sharp on choosing $c_2 = p_1 = 2i$, $c_2 = p_2 = -2$, $b_2 = 2i(1-\beta)$ and $b_3\ (3-2\beta)(\beta-1)$ and substituting in (10).

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